Function Spaces
Assignment - III

1. Define P_n recursively by $P_{n+1}(x) = P_n(x) + \frac{x - P_n(x)^2}{2}$, where $P_0(x) = 0$.
   (a) Show that each P_n is a polynomial.
   (b) Check that $0 \leq P_n(x) \leq P_{n+1}(x) \leq \sqrt{x}$ for $x \in [0,1]$. Using Dini's theorem, prove that $\{P_n\}_{n\geq 0}$ converges to $\sqrt{x}$ uniformly on $[0,1]$.
   (c) Show that $P_n(x^2)$ is also a polynomial, and $\{P_n(x^2)\}_{n\geq 0}$ converges to $|x|$ uniformly on $[-1,1]$. Therefore, there is a sequence of polynomials $\{P_n\}_{n\geq 0}$ that converges uniformly to $|x|$ on $[-1,1]$.

2. If $f \in C[a,b]$, and $\int_a^b x^n f(x)dx = 0$ for all $n \geq 0$, then prove that $f = 0$.

3. Let $\{x_i\}_{i\geq 1}$ be a sequence of numbers in $(0,1)$ such that $\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^n x_i^k$ exists for all $k \geq 0$. Show that $\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^n f(x_i)$ exists for all $f \in C[0,1]$.

4. Construct a sequence of polynomials that converge uniformly on $[0,1]$ but whose derivatives fail to converge uniformly on $[0,1]$.

5. Use the Weierstrass approximation theorem to prove each of the following statements.
   (a) If f is continuous on $[1,+\infty)$ and if $f(x) \to a$ as $x \to +\infty$, then f can be uniformly approximated on $[1,+\infty)$ by a function g of the form $g(x) = p(\frac{1}{x})$, where p is a polynomial.
   (b) If f is continuous on $[0,+\infty)$ and if $f(x) \to a$ as $x \to +\infty$, then f can be uniformly approximated on $[0,+\infty)$ by a function g of the form $g(x) = p(e^{-x})$ where p is a polynomial.

6. We denote by $\mathrm{Lip}_K\alpha$, the set of functions $f \in C[0,1]$ that are Lipschitz of order α with constant K on $[0,1]$, where $0 < \alpha \leq 1$ and $0 < K < \infty$. That is,

$$\mathrm{Lip}_K\alpha := \big\{f : [0,1] \to \mathbb{R} : |f(x) - f(y)| \leq K|x-y|^\alpha,\ x, y \in [0,1]\big\}.$$

   Define $\mathrm{Lip}\,\alpha = \bigcup_{K=1}^\infty \mathrm{Lip}_K\alpha$.
   (a) Show that $\mathrm{Lip}_K\alpha$ is closed in $C[0,1]$. In fact, if a sequence $\{f_n\}_{n\geq 1}$ in $\mathrm{Lip}_K\alpha$ converges pointwise to f on $[0,1]$, show that $f \in \mathrm{Lip}_K\alpha$. Is $\mathrm{Lip}_K\alpha$ a subspace of $C[0,1]$?
   (b) Show that $\mathrm{Lip}\,\alpha$ is a subspace of $C[0,1]$. Is $\mathrm{Lip}\,\alpha$ a subalgebra of $C[0,1]$?

7. (a) Let f be 2π-periodic and Riemann integrable. Then show that

$$\lim_{x\to 0}\int_{-\pi}^{\pi} |f(x+t) - f(t)|^2 dt = 0.$$

   (b) Let $f, k \in C^{2\pi}$, the space of all 2π-periodic continuous functions $f : \mathbb{R} \to \mathbb{R}$. Then prove that $g \in C^{2\pi}$, where

$$g(x) = \int_{-\pi}^{\pi} f(x+t)k(t)dt.$$

(c) Assume only that f is 2π-periodic and Riemann integrable and $k \in C^{2\pi}$. Is g (defined above) continuous?

(d) We assume that f, k is 2π-periodic and Riemann integrable. Is g (defined above) still continuous?

8. Let $\{k_n\}_{n\geq 1}$ be a positive summability kernel. Then show that if f is Riemann integrable, then

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)k_n(t)dt \to f(x) \quad \text{pointwise, as } n \to \infty, \quad \text{at each point of continuity of } f.$$

9. Prove that differentiability of f at a point implies convergence of its Fourier series at the point.

10. Using Fourier series of appropriate functions, prove the following identities.

(a)

$$x = 2\sum_{n=1}^{\infty} \frac{(-1)^{n-1}\sin nx}{n}, \quad (-\pi < x < \pi).$$

(b)

$$x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n \cos nx}{n^2}, \quad (-\pi \leq x \leq \pi).$$

(c)

$$\frac{\pi}{4} = \sum_{n=1}^{\infty} \frac{\sin{(2n-1)x}}{2n-1}, \quad (0 < x < \pi).$$