

Function Spaces

Assignment - II

1. Let (M, d) be a metric space. Suppose that $f, f_n : M \rightarrow \mathbb{R}$ ($n \geq 1$).

(a) Show that the set on which $\{f_n\}_{n \geq 1}$ converges pointwise to f is given by

$$\bigcap_{k=1}^{\infty} \bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} \{x \in M : |f_n(x) - f(x)| \leq \frac{1}{k}\}.$$

(b) What is the set on which $\{f_n(x)\}_{n \geq 1}$ is Cauchy? If each f_n is continuous on M , what type of set is this?

2. Let $f_n : [a, b] \rightarrow \mathbb{R}$ satisfy $|f_n(x)| \leq 1$ for all $x \in [a, b]$ and $n \geq 1$. Show that there is a subsequence $\{f_{n_k}\}_{k \geq 1}$ such that $\lim_{k \rightarrow \infty} f_{n_k}(x)$ exists for each rational $x \in [a, b]$.

3. (a) A collection of real-valued functions \mathcal{F} on M is said to be pointwise bounded if, for each $x \in M$, the set $\{f(x) : f \in \mathcal{F}\}$ is bounded (in \mathbb{R}), that is, if $\sup_{f \in \mathcal{F}} |f(x)|$ for each $x \in M$. If $\{f_n\}_{n \geq 1}$ is a pointwise convergent sequence of real-valued functions, show that $\{f_n\}_{n \geq 1}$ is also pointwise bounded.

(b) Prove that a uniformly bounded collection of functions is also pointwise bounded. Give an example of a collection of functions that is pointwise bounded but not uniformly bounded.

4. (a) Show that $\sum_{n=1}^{\infty} \frac{x^2}{(1+x^2)^n}$ converges for all $|x| \leq 1$, but it does not converge uniformly on $|x| \leq 1$.

(b) Show that both $\sum_{n=1}^{\infty} x^n(1+x^n)$ and $\sum_{n=1}^{\infty} (-1)^n x^n(1+x^n)$ are convergent on $[0, 1]$, but only one converges uniformly. which one? why?

5. Let (M, d) be a compact metric space, and let $\{f_n\}_{n \geq 1}$ be an equicontinuous sequence in $C(M)$. If $\{f_n\}_{n \geq 1}$ is pointwise convergent, prove that, in fact, $\{f_n\}_{n \geq 1}$ is uniformly convergent.

6. Let (M, d) be a compact metric space, and let $\{f_n\}_{n \geq 1}$ be an equicontinuous sequence in $C(M)$. Show that

$$\mathcal{S} = \{x \in M : \{f_n(x)\}_{n \geq 1} \text{ converges}\}$$

is a closed set in M .

7. If $\{f_n\}_{n \geq 1}$ is an equicontinuous sequence in $C[a, b]$, and if $\{f_n(x)\}_{n \geq 1}$ converges at each rational in $[a, b]$, prove that it is uniformly convergent on $[a, b]$.

8. (a) Define $T : C[a, b] \rightarrow C[a, b]$ by

$$(Tf)(x) = \int_a^x f(t) dt.$$

Show that T maps bounded sets into equicontinuous (and hence compact) sets.

(b) Let $\{f_n\}_{n \geq 1}$ be a sequence in $C[a, b]$ with $\|f_n\|_{\infty} \leq 1$ for all n , and define

$$F_n(x) = \int_a^x f_n(t) dt.$$

Show that some subsequence of $\{F_n\}_{n \geq 1}$ is uniformly convergent.

9. Let $K(x, t)$ be a continuous function on the square $[a, b] \times [a, b]$.

(a) Given $f \in C[a, b]$, show that

$$g(x) = \int_a^b f(t)K(x, t) dt$$

defines a continuous function in $C[a, b]$.

(b) Define $T : C[a, b] \rightarrow C[a, b]$ by

$$(Tf)(x) = \int_a^b f(t)K(x, t) dt.$$

Show that T maps bounded sets into equicontinuous sets. In particular, T is continuous.

10. Suppose that $F : \mathbb{R}^2 \rightarrow \mathbb{R}$ is continuous and Lipschitz in its second variable, that is,

$$|F(r, s) - F(r, t)| \leq k|s - t| \quad (r, s, t \in \mathbb{R}).$$

(a) Given $f \in C[a, b]$, show that

$$g(x) = \int_a^x F(t, f(t)) dt$$

defines a continuous function in $C[a, b]$.

(b) Define $T : C[a, b] \rightarrow C[a, b]$ by

$$(Tf)(x) = \int_a^x F(t, f(t)) dt.$$

Show that T maps bounded sets into equicontinuous sets. In particular, T is continuous.