# Function Spaces
# Assignment - 1

1. Answer the following problems with proper justification:

   (a) Is the set $\{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 - x_3^2 = 1\}$ compact set in $\mathbb{R}^3$?

   (b) Let $M_n(\mathbb{R})$ be the space of $n \times n$ matrices over $\mathbb{R}$ and think as $M_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$. Then

      i. Is $GL(n, \mathbb{R})$, the set of all invertible matrices, in $M_n(\mathbb{R})$ compact?

      ii. Is $O(n, \mathbb{R})$, the set of all orthogonal matrices, in $M_n(\mathbb{R})$ compact?

      iii. Is $SL(n, \mathbb{R})$, the set of all matrices with determinant 1, in $M_n(\mathbb{R})$ compact?

      iv. Is $N(n, \mathbb{R})$, the set of all nilpotent matrices, in $M_n(\mathbb{R})$ compact?

2. If $f : (0, 1) \to \mathbb{R}$ is uniformly continuous, then show that $\lim_{x \to 0} f(x)$ exists. Conclude that f is bounded on (0, 1 ).

3. Let $K$ be a compact subset of a metric space $(M, d)$, and let $G$ be an open set containing $K$. Then prove that there exists an open set $U$ for which $K \subseteq U \subseteq \overline{U} \subseteq G$.

4. Let $M$ be compact and let $f : M \to M$ satisfy $d(f(x), f(y)) = d(x, y)$ for all $x, y \in M$, Show that $f$ is one-one and onto.

5. Suppose $f$ is a real-valued continuous function on $(\mathbb{R}^n, \|.\|_2)$ with the property that there is a number $c > 0$ such that $|f(x)| \geq c\|x\|_2$ for all $x \in \mathbb{R}$. Show that if K is a compact subset of $\mathbb{R}$, then $f^{-1}(K)$ is compact set in $\mathbb{R}^n$.

6. A real-valued function $f$ on a metric space $(M, d)$ is said to be lower semi-continuous if for each $\alpha \in \mathbb{R}$, the set $\{x \in M : f(x) > \alpha\}$ is open in $M$. Prove that if M is compact, then every lower semi-continuous function on $M$ is bounded below and attains a minimum value.

7. Consider the normed linear space (NLS)

$$l_\infty := \{x = (x_1, x_2, \ldots, ) : x_n \in \mathbb{R}, n \geq 1, x \text{ is a bounded sequence}\}$$

   with the norm $\|x\|_\infty = \sup_{n \geq 1} |x_n|$. Let $K = \{x = (x_1, x_2, \ldots, ) \in l_\infty : \lim_{n \to \infty} x_n = 1\}$. Prove that

(a) K is a closed subset of $l_\infty$.

(b) If $T : l_\infty \to l_\infty$ is defined by $T(x) = (0, x_1, x_2, \ldots)$ for $x = (x_1, x_2, \ldots) \in l_\infty$, that is, if $T$ shifts the entries forward and puts 0 in the empty slot, then $T(K) \subseteq K$.

(c) $T$ is an isometry on $K$, but $T$ has no fixed point in $K$.

8. Show that the map $T$ on $(C[0,1], \|.\|_\infty)$ defined by

$$Tf(x) := \int_0^x (x-t)f(t)\mathrm{dt}, \quad 0 \le x \le 1, \quad f \in C[0,1].$$

is a contraction. What is its fixed point?

9. Let $a, b$ be real numbers with $0 < b < 1$. Consider the subset $X$ of $C[0,b]$ consisting of functions $f$ such that $f(0) = a$. Then show that $X$ is closed in $(C[0,b], \|.\|_\infty)$. Define

$$Tf(x) := a + \int_0^x |f(t)|\mathrm{dt} \quad (0 \le x \le b).$$

Prove that there exists a unique $f \in X$ which satisfies $f' = |f|$ on $(0,b)$.

10. Let $a > 0$ and $g \in C[0,a]$. Define $T : C[0,a] \to C[0,a]$ as follows:

$$Tf(x) := \int_0^x f(t)dt + g(x), \quad (f \in C[0,a],\ 0 \le x \le a).$$

Show that $T$ is a contraction on $(C[0,a], \|.\|_\infty)$ if and only if $a < 1$. Assume that $g$ is differentiable. Find the initial value problem of an ordinary differential equation whose solution is the fixed point of $T$.